12.3) Angles Between Vectors And The Dot Product

1. Angles Between Vectors:

Given any two nonzero vectors in the same dimensional space (either both two-dimensional or both three-dimensional), we can identify a unique **angle** between these two vectors. To visualize this angle, we should depict the vectors as two directed line segments with a common initial point.

Since the zero vector, $\mathbf{0}$, cannot be represented as a directed line segment, the angle between $\mathbf{0}$ and any other vector is **undefined**.

Angles between vectors can be measured in either *radians* or *degrees.* If a measurement is stated without the units being specified, then radians are assumed.

Angles between vectors differ from angles as studied in Trig class, in two important regards:

1. In Trig class, every nonzero angle has an *orientation*, which is either *clockwise* or *counter-clockwise*. Counter-clockwise angles have *positive* measurements, while clockwise angles have *negative* measurements. In vector theory, however, angles have *no orientation*, so we will never deal with negative angle measurements. The measure of an angle between vectors is always *greater than or equal to* 0 radians or 0 degrees.
2. In Trig class, an angle can have a measure greater than π radians or 180 degrees. In vector theory, this is not possible. The measure of an angle between vectors is always *less than or equal to* π radians or 180 degrees.

If two nonzero vectors have the *same direction*, then the angle between them is 0 radians or 0 degrees.

If two nonzero vectors have *opposite direction*, then the angle between them is π radians or 180 degrees.

Two nonzero vectors are said to be **parallel** if and only if they have either the same direction or opposite direction–i.e., if and only if the angle between them is either 0 or π. (The zero vector cannot be considered parallel to any vector, because it has no direction.)

Two nonzero vectors are parallel if and only if each is a nonzero scalar multiple of the other. (If the scalar is positive, the vectors have the same direction; if the scalar is negative, the vectors have opposite direction.)

Example One: Let $\mathbf{a} = < 2, -3 >$, $\mathbf{b} = < 4, -6 >$, and $\mathbf{c} = < -6, 9 >$. Note that $\mathbf{b} = 2\mathbf{a}$, $\mathbf{c} = -3\mathbf{a}$, and $\mathbf{c} = -1.5\mathbf{b}$. The three vectors are pairwise parallel. $\mathbf{a}$ and $\mathbf{b}$ have the same direction, but $\mathbf{a}$ and $\mathbf{c}$ have opposite direction, as do $\mathbf{b}$ and $\mathbf{c}$.

If two nonzero vectors are *not* scalar multiples of each other, then they are **nonparallel** and the angle between them is *between* 0 and π.

If the angle between two nonzero vectors is $\frac{\pi}{2}$ radians or 90 degrees, the two vectors are said to be **perpendicular**.

Example Two: In the x,y plane, the vectors $\mathbf{i} = < 1,0 >$ and $\mathbf{j} = < 0,1 >$ are perpendicular. In x,y,z space, the vectors $\mathbf{i} = < 1,0,0 >$, $\mathbf{j} = < 0,1,0 >$, and $\mathbf{k} = < 0,0,1 >$ are pairwise perpendicular.

An angle of $\frac{\pi}{2}$ is said to be a **right angle**, an angle of π is said to be a **straight angle**, an angle of 0 is said to be a **zero angle**, an angle between 0 and $\frac{\pi}{2}$ is said to be an **acute angle**, and an angle between $\frac{\pi}{2}$ and π is said to be an **obtuse angle**. Hence, if two nonzero vectors are parallel, then the angle between them is either zero or straight, and if they are nonparallel, then the angle between them is either acute, right, or obtuse.

Given two nonzero vectors in component form, how can we calculate the angle between them? We shall develop a formula for this calculation, but first we must introduce the concept of the "dot product" of two vectors.

2. The Dot Product and Orthogonal Vectors:

Given any two vectors $\mathbf{a} = < a_1, a_2 >$ and $\mathbf{b} = < b_1, b_2 >$, their **dot product** is $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$.

Given any two vectors $\mathbf{a} = < a_1, a_2, a_3 >$ and $\mathbf{b} = < b_1, b_2, b_3 >$, their **dot product** is $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Note that the result of the dot product is a *scalar*. It can be positive or negative or zero.

Example Three: $< 2,7 > \cdot < 3,-4 > = (2)(3) + (7)(-4) = 6 - 28 = -22$.

Example Four: $< 5,-1,3 > \cdot < 7,-2,-6 > = (5)(7) + (-1)(-2) + (3)(-6) = 35 + 2 - 18 = 19$.

In the x,y plane, if $\mathbf{a} = < a_1, a_2 >$, then $\mathbf{a} \cdot \mathbf{i} = a_1$ and $\mathbf{a} \cdot \mathbf{j} = a_2$.

In x,y,z space, if $\mathbf{a} = < a_1, a_2, a_3 >$, then $\mathbf{a} \cdot \mathbf{i} = a_1$ and $\mathbf{a} \cdot \mathbf{j} = a_2$ and $\mathbf{a} \cdot \mathbf{k} = a_3$.

Recall from basic arithmetic that a *compound expression* is one involving more than one operation (or the same operation carried out multiple times). We must be very careful when we write compound expressions involving vector operations. If we're not careful, we could write an expression that makes no sense. For instance, the expression $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$ makes no sense, because the first dot product produces a scalar, so the second dot product cannot be carried out (the dot product can only be performed on two vectors). The expressions

$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ and $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c})$ make sense, but the expressions $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ and $\mathbf{a} \cdot \mathbf{b} - \mathbf{c}$ do not (the dot product produces a scalar, and we cannot add or subtract a scalar and a vector). In expressions such as $\mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{d}$ or $\mathbf{a} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{d}$, the plus sign and the minus sign are understood to represent *scalar addition and subtraction*, rather than *vector addition and subtraction*, and it is understood that the dot products are carried out first.

The dot product has the following properties:

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (The dot product is commutative)
- $c(\mathbf{a} \cdot \mathbf{b}) = (c\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{b})$ (We may omit parentheses and write $c\mathbf{a} \cdot \mathbf{b}$.)
- $(c\mathbf{a}) \cdot (d\mathbf{b}) = cd(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{0} \cdot \mathbf{a} = 0$, for any vector $\mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = a^2$, and $a = \sqrt{\mathbf{a} \cdot \mathbf{a}}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
- $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c}$

The last four properties can be summarized as follows: The dot product distributes over vector addition and subtraction.

On the basis of the above distributive properties, the product of vector binomials can be distributed out, just like products of ordinary binomials (a process commonly known as FOIL).

Example Five: $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{d}$

With scalar multiplication, if $xy = 0$, then either $x = 0$ or $y = 0$. This does not work with vectors! If $\mathbf{a} \cdot \mathbf{b} = 0$, we *cannot* infer that $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$. It is possible that two *nonzero* vectors can have a dot product of 0.

Example Six: The dot product of $< 3,5 >$ and $< 5,-3 >$ is 0.

Example Seven: The dot product of $< 2,9,-5 >$ and $< 6,7,15 >$ is 0.

If two vectors have a dot product of 0, then they are said to be **orthogonal** to each other (more briefly, we just say they "are orthogonal"). Thus, $< 3,5 >$ and $< 5,-3 >$ are orthogonal, as are $< 2,9,-5 >$ and $< 6,7,15 >$.

$\mathbf{0}$ is orthogonal to every vector.

Given any nonzero vector, there are infinitely many nonzero vectors orthogonal to it.

- Given any nonzero vector $< p,q >$, every scalar multiple of $< q,-p >$ or $< -q,p >$ is orthogonal to it. Furthermore, *every* vector orthogonal to $< p,q >$ must be a scalar multiple of $< q,-p >$ or $< -q,p >$.

- Given any nonzero vector $< p,q,r >$, every scalar multiple of each of the following vectors is orthogonal to it:
  $< q,-p,0 >$ or $< -q,p,0 >$
  $< r,0,-p >$ or $< -r,0,p >$
  $< 0,r,-q >$ or $< 0,-r,q >$
  This does *not* give us an exhaustive account of all vectors orthogonal to $< p,q,r >$.

Example Eight: The following vectors are orthogonal to $< 4,-7 >\ldots$
- $< -7,-4 >$
- $< 7,4 >$
- $< 63,36 >$
- $< -84,-48 >$
- $< \frac{7}{13},\frac{4}{13} >$
- $< 1,\frac{4}{7} >$
- $< \frac{7}{4},1 >$

Example Nine: The following vectors are orthogonal to $< 3,6,5 >\ldots$
- $< 6,-3,0 >$
- $< 5,0,-3 >$
- $< 0,5,-6 >$
- $< -12,6,0 >$
- $< 1.25,0,-0.75 >$
- $< 0,30,-36 >$
- $< 1,-\frac{1}{2},0 >$

3. The Relationship Between the Dot Product and Angle Measure:

The Dot Product Theorem: For any two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, if θ is the angle between them, then $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$.

Proof:

First, suppose $\mathbf{a}$ and $\mathbf{b}$ have the same direction ($\theta = 0$). Since $\cos 0 = 1$, we must show that $\mathbf{a} \cdot \mathbf{b} = ab$. Because our vectors are in the same direction, there exists a positive scalar c such that $\mathbf{a} = c\mathbf{b}$, so $a = cb$. $\mathbf{a} \cdot \mathbf{b} = (c\mathbf{b}) \cdot \mathbf{b} = c(\mathbf{b} \cdot \mathbf{b}) = cb^2$. $ab = (cb)b = cb^2$. Thus, $\mathbf{a} \cdot \mathbf{b} = ab$.

Second, suppose $\mathbf{a}$ and $\mathbf{b}$ have opposite direction ($\theta = \pi$). Since $\cos\pi = -1$, we must show that $\mathbf{a} \cdot \mathbf{b} = -ab$. Because our vectors are in the opposite direction, there exists a negative scalar c such that $\mathbf{a} = c\mathbf{b}$, so $a = |c|b = -cb$. $\mathbf{a} \cdot \mathbf{b} = (c\mathbf{b}) \cdot \mathbf{b} = c(\mathbf{b} \cdot \mathbf{b}) = cb^2$. $ab = (-cb)b = -cb^2$, so $-ab = cb^2$. Thus, $\mathbf{a} \cdot \mathbf{b} = -ab$.

Third, suppose $\mathbf{a}$ and $\mathbf{b}$ are nonparallel. Place them so they have a common tail. Place $\mathbf{a} - \mathbf{b}$ so that its tail is at the tip of $\mathbf{b}$ and its tip is at the tip of $\mathbf{a}$. The three vectors now form a triangle, with sides of lengths a, b, and $|\mathbf{a} - \mathbf{b}|$. By the Law of Cosines, $|\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab\cos\theta$.

$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2$.

Thus, $a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2 = a^2 + b^2 - 2ab\cos\theta$.

Subtracting a^2 and b^2 from both sides gives us $-2\mathbf{a} \cdot \mathbf{b} = -2ab\cos\theta$.

Dividing both sides by -2 gives us $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$.

End of Proof (**QED**).

Corollary 1 to the Dot Product Theorem: Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$.

- $\mathbf{a}$ and $\mathbf{b}$ are in the same direction ($\theta = 0$) iff $\cos\theta = 1$ iff $\mathbf{a} \cdot \mathbf{b} = ab$.
- $\mathbf{a}$ and $\mathbf{b}$ are in the opposite direction ($\theta = \pi$) iff $\cos\theta = -1$ iff $\mathbf{a} \cdot \mathbf{b} = -ab$.
- $\mathbf{a}$ and $\mathbf{b}$ are parallel iff $\cos\theta = \pm 1$ iff $\cos^2\theta = 1$ iff $\sin\theta = 0$ iff $\mathbf{a} \cdot \mathbf{b} = \pm ab$.

Corollary 2 to the Dot Product Theorem: For any two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, if θ is the angle between them, then $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab}$, and $\theta = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{ab}$.

Note that the inverse cosine always produces an angle in the interval $[0, \pi]$, which is exactly what we want. Also note that for any two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, $\frac{\mathbf{a}\cdot\mathbf{b}}{ab} \in [-1, 1]$, which is precisely the domain of the inverse cosine function.

Since $\theta = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{ab}$, it follows immediately that:

- $\theta = \frac{\pi}{2}$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$
- $\theta < \frac{\pi}{2}$ if and only if $\mathbf{a} \cdot \mathbf{b} > 0$
- $\theta > \frac{\pi}{2}$ if and only if $\mathbf{a} \cdot \mathbf{b} < 0$

(You should confirm this by examining the graph of the inverse cosine function. Bear in mind that ab is positive, so the sign of $\frac{\mathbf{a}\cdot\mathbf{b}}{ab}$ is determined by the sign of $\mathbf{a} \cdot \mathbf{b}$.)

Two nonzero vectors are perpendicular if and only if they are orthogonal. Thus, for nonzero vectors, “perpendicular” and “orthogonal” are essentially synonymous. However, this connection breaks down when we consider the zero vector. $\mathbf{0}$ is orthogonal to every vector, but we cannot call it perpendicular to any vector.

4. Parallel Lines and Vectors, and Angles Between Lines:

A line and a nonzero vector are said to be **parallel** if the vector has a representation where both the initial point and the terminal point lie on the line. If this is the case, then any representation of the vector that intersects the line must be a subset of the line (i.e., if *any* point of the representation lies on the line, then *every* point of the representation lies on the line).

If a nonzero vector $\mathbf{a}$ is parallel to a line, then its opposite, $-\mathbf{a}$, is also parallel to that line.

Example 10: In the x,y plane, the line $y = 2x + 3$ and the vector $< 1,2 >$ are parallel, because one of the vector's representation has initial point $(0,3)$ and terminal point $(1,5)$, both of which points lie on the line. Likewise, the vector $< -1,-2 >$ is parallel to this line, because one of the vector's representation has initial point $(0,3)$ and terminal point $(-1,1)$, both of which points lie on the line.

In the x,y plane, a line is *vertical* if and only if it is parallel to $\mathbf{j} = < 0,1 >$. In x,y,z space, a line is *vertical* if and only if it is parallel to $\mathbf{k} = < 0,0,1 >$.

In the x,y plane, a line is *horizontal* if and only if it is parallel to $\mathbf{i} = < 1,0 >$. In x,y,z space, a line is *horizontal* if and only if it is parallel to a nonzero linear combination of $\mathbf{i} = < 1,0,0 >$ and $\mathbf{j} = < 0,1,0 >$.

Example 11: In x,y,z space, any line parallel to $\mathbf{i}$ or $\mathbf{j}$ or $\mathbf{i} + \mathbf{j}$ or $\mathbf{i} - \mathbf{j}$ is horizontal.

In the x,y plane, two nonvertical lines are parallel if and only if they have the same slope. Any two vertical lines are parallel and have undefined slope. (A reference to "same" slope implies the existence of slope, so we would not say that vertical lines have the same slope.) For any real number m, a line has slope m if and only if it is parallel to the vector $< 1,m >$. Note that the length of this vector is $\sqrt{1 + m^2}$.

Example 12: In the x,y plane, the line $y = -\frac{3}{5}x + 7$ is parallel to the vector $< 1,-\frac{3}{5} >$.

Let L_1 and L_2 be two distinct and nonparallel lines in the x,y plane. These lines intersect at exactly one point. We can identify *two angles* between these lines, which are *supplements* of each other (i.e., the two angles add up to π radians or 180 degrees); for example, the angles could be 35° and 145°. At least one of the lines must be nonvertical. Suppose L_1 is nonvertical. Let m_1 be the slope of L_1 (so L_1 is parallel to $< 1,m_1 >$).

- If L_2 is vertical, then L_2 is parallel to $\mathbf{j} = < 0,1 >$. One of the angles between L_1 and L_2 is the angle between $\mathbf{j}$ and $< 1,m_1 >$, which is $\cos^{-1}\frac{m_1}{\sqrt{1 + (m_1)^2}}$.
- If L_2 is nonvertical, then L_2 has slope m_2 and is parallel to $< 1,m_2 >$. One of the angles between L_1 and L_2 is the angle between $< 1,m_1 >$ and $< 1,m_2 >$, which is $\cos^{-1}\frac{1 + m_1 m_2}{\sqrt{1 + (m_1)^2}\sqrt{1 + (m_2)^2}}$.

Example 13: In the x, y plane, let L_1 be the line $6x - 5y = 2$, which has slope $m_1 = \frac{6}{5}$, and let L_2 be the line $8x + 3y = -9$, which has slope $m_2 = -\frac{8}{3}$. These lines are parallel to the vectors $\mathbf{u}_1 = < 1, \frac{6}{5} >$ and $\mathbf{u}_2 = < 1, -\frac{8}{3} >$, respectively, whose respective lengths are $\sqrt{1 + \frac{36}{25}}$ and $\sqrt{1 + \frac{64}{9}}$. The angle between these vectors is $\cos^{-1} \frac{1 - \frac{48}{15}}{\sqrt{1 + \frac{36}{25}} \sqrt{1 + \frac{64}{9}}} = \cos^{-1} \frac{-33}{\sqrt{4{,}453}}$, which is about 2.088 radians or $119.6°$. Thus, the angles between our two lines are about 2.088 and 1.054 radians, or $119.6°$ and $60.4°$.

5. Vector Direction:

In Section 12.2, we introduced the idea of a nonzero vector's *direction*, but we did not develop this idea rigorously. We are now in a position to do so. Vector direction is pinpointed by the specification of various angles.

In two-dimensional space, *two* angles are required to uniquely determine a vector's direction, since only one angle would be ambiguous (since vector angles are non-oriented, two different directions can be associated with just one angle). Similarly, *three* angles are required in three-dimensional space.

For any nonzero vector $\mathbf{a}$ in two-dimensional space, its direction is determined by the angle α between $\mathbf{a}$ and $\mathbf{i} = < 1, 0 >$, and by the angle β between $\mathbf{a}$ and $\mathbf{j} = < 0, 1 >$. We refer to α and β as the **direction angles** of $\mathbf{a}$. If $\mathbf{a} = < a_1, a_2 >$, then $\alpha = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{i}}{a(1)} = \cos^{-1} \frac{a_1}{a}$, and $\beta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{j}}{a(1)} = \cos^{-1} \frac{a_2}{a}$. Equivalently, $\cos \alpha = \frac{a_1}{a}$ and $\cos \beta = \frac{a_2}{a}$. $\cos \alpha$ and $\cos \beta$ are referred to as the **direction cosines** of the vector $\mathbf{a}$. Recall that the unit vector in the direction of $\mathbf{a}$ is $\frac{\mathbf{a}}{a} = < \frac{a_1}{a}, \frac{a_2}{a} >$. Thus, $\cos \alpha$ and $\cos \beta$ are, respectively, the first and second components of the unit vector in the direction of $\mathbf{a}$. $a_1 = a \cos \alpha$ and $a_2 = a \cos \beta$, so $\mathbf{a} = < a \cos \alpha, a \cos \beta > = a < \cos \alpha, \cos \beta >$. Since $\frac{\mathbf{a}}{a} = < \cos \alpha, \cos \beta >$, it follows that $\cos^2 \alpha + \cos^2 \beta = 1$. This is known as the **Pythagorean Identity.**

For any nonzero vector $\mathbf{a}$ in three-dimensional space, its direction is determined by the angle α between $\mathbf{a}$ and $\mathbf{i} = < 1, 0, 0 >$, and by the angle β between $\mathbf{a}$ and $\mathbf{j} = < 0, 1, 0 >$, and by the angle γ between $\mathbf{a}$ and $\mathbf{k} = < 0, 0, 1 >$. We refer to α, β, and γ as the **direction angles** of $\mathbf{a}$. If $\mathbf{a} = < a_1, a_2, a_3 >$, then $\alpha = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{i}}{a(1)} = \cos^{-1} \frac{a_1}{a}$, and $\beta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{j}}{a(1)} = \cos^{-1} \frac{a_2}{a}$, and $\gamma = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{k}}{a(1)} = \cos^{-1} \frac{a_3}{a}$. Equivalently, $\cos \alpha = \frac{a_1}{a}$ and $\cos \beta = \frac{a_2}{a}$ and $\cos \gamma = \frac{a_3}{a}$. $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are referred to as the **direction cosines** of the vector $\mathbf{a}$. Recall that the unit vector in the direction of $\mathbf{a}$ is $\frac{\mathbf{a}}{a} = < \frac{a_1}{a}, \frac{a_2}{a}, \frac{a_3}{a} >$. Thus, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are, respectively, the first, second, and third components of the unit vector in the direction of $\mathbf{a}$. $a_1 = a \cos \alpha$ and $a_2 = a \cos \beta$ and $a_3 = a \cos \gamma$, so $\mathbf{a} = < a \cos \alpha, a \cos \beta, a \cos \gamma >$ $= a < \cos \alpha, \cos \beta, a \cos \gamma >$. Since $\frac{\mathbf{a}}{a} = < \cos \alpha, \cos \beta, \cos \gamma >$, it follows that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. This is known as the **Pythagorean Identity**.

Example 14: Here is a table of two-dimensional vectors and their direction angles (measured in degrees). Each vector is a unit vector.

Vector:	α	β
$< 1, 0 > = \mathbf{i}$	0°	90°
$< \frac{\sqrt{3}}{2}, \frac{1}{2} >$	30°	60°
$< \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >$	45°	45°
$< \frac{1}{2}, \frac{\sqrt{3}}{2} >$	60°	30°
$< 0, 1 > = \mathbf{j}$	90°	0°
$< -\frac{1}{2}, \frac{\sqrt{3}}{2} >$	120°	30°
$< -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >$	135°	45°
$< -\frac{\sqrt{3}}{2}, \frac{1}{2} >$	150°	60°
$< -1, 0 > = -\mathbf{i}$	180°	90°
$< -\frac{\sqrt{3}}{2}, -\frac{1}{2} >$	150°	120°
$< -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} >$	135°	135°
$< -\frac{1}{2}, -\frac{\sqrt{3}}{2} >$	120°	150°
$< 0, -1 > = -\mathbf{j}$	90°	180°
$< \frac{1}{2}, -\frac{\sqrt{3}}{2} >$	60°	150°
$< \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} >$	45°	135°
$< \frac{\sqrt{3}}{2}, -\frac{1}{2} >$	30°	120°

Example 15: Let $\mathbf{a} = < 3, -5, 6 >$. $a = \sqrt{70}$. The direction angles of $\mathbf{a}$ are $\alpha = \cos^{-1}\frac{3}{\sqrt{70}} \approx 1.204$, and $\beta = \cos^{-1}\frac{-5}{\sqrt{70}} \approx 2.211$, and $\gamma = \cos^{-1}\frac{6}{\sqrt{70}} \approx 0.771$.

Two nonzero vectors have the **same direction** if and only if their corresponding direction angles are *equal*, and they have **opposite direction** if and only if their corresponding direction angles are *supplementary*. In other words:

- In the x,y plane, suppose $\mathbf{a}$ has direction angles α_1 and β_1, and suppose $\mathbf{b}$ has direction angles α_2 and β_2.
  $\mathbf{a}$ and $\mathbf{b}$ have the same direction if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.
  $\mathbf{a}$ and $\mathbf{b}$ have opposite direction if and only if $\alpha_1 + \alpha_2 = \pi$ and $\beta_1 + \beta_2 = \pi$.
- In x,y,z space, suppose $\mathbf{a}$ has direction angles α_1, β_1, and γ_1, and suppose $\mathbf{b}$ has direction angles α_2, β_2, and γ_2.
  $\mathbf{a}$ and $\mathbf{b}$ have the same direction if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$.
  $\mathbf{a}$ and $\mathbf{b}$ have opposite direction if and only if $\alpha_1 + \alpha_2 = \pi$ and $\beta_1 + \beta_2 = \pi$ and $\gamma_1 + \gamma_2 = \pi$.

In Section 12.2, we stated, but did not prove, that any nonzero vector and its additive inverse have opposite directions. We are now in a position to prove this. We shall do so for a two-dimensional vector; the three-dimensional proof is analogous. Let $\mathbf{a} = < a_1, a_2 >$ be a nonzero vector. Its additive inverse is $-\mathbf{a} = < -a_1, -a_2 >$. Let α_1 and β_1 be the direction angles of $\mathbf{a}$, and let α_2 and β_2 be the direction angles of $-\mathbf{a}$. $\alpha_1 = \cos^{-1}\frac{a_1}{a}$, and $\beta_1 = \cos^{-1}\frac{a_2}{a}$. $\alpha_2 = \cos^{-1}\frac{-a_1}{a}$, and $\beta_1 = \cos^{-1}\frac{-a_2}{a}$. For any $x \in [-1, 1]$, $\cos^{-1}x + \cos^{-1}(-x) = \pi$. Therefore $\alpha_1 + \alpha_2 = \pi$ and $\beta_1 + \beta_2 = \pi$.

6. **Vector Projections and Work**:

Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero vectors in either two dimensions or in three dimensions. The **vector projection** of $\mathbf{b}$ **onto** $\mathbf{a}$ is the vector $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$, or $\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$. This is denoted $proj_{\mathbf{a}}\mathbf{b}$.

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

If $\theta = \frac{\pi}{2}$ (i.e., if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular), then $\mathbf{a}\cdot\mathbf{b} = 0$ and $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a} = 0\mathbf{a} = \mathbf{0}$.

If $\theta = 0$ or $\theta = \pi$ (i.e., if $\mathbf{a}$ and $\mathbf{b}$ are parallel), then there exists a nonzero scalar c such that $\mathbf{b} = c\mathbf{a}$. In this case, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(c\mathbf{a}) = c(\mathbf{a}\cdot\mathbf{a}) = ca^2$, so $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2} = \frac{ca^2}{a^2} = c$, and $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a} = c\mathbf{a} = \mathbf{b}$.

If θ is acute or obtuse, then $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$ can be understood in terms of triangle geometry. For this purpose, let us consider representations of $\mathbf{a}$ and $\mathbf{b}$ having a common tail, P. Let A be the tip of $\mathbf{a}$, and let B be the tip of $\mathbf{b}$, so we have the directed line segments $\overrightarrow{PA}$ for $\mathbf{a}$ and $\overrightarrow{PB}$ for $\mathbf{b}$. Let ℓ be the line through B that is perpendicular to the line $\overleftrightarrow{PA}$, and let Q be the point of intersection between ℓ and $\overleftrightarrow{PA}$. If θ is acute, then Q lies on the same side of P as A, but if θ is obtuse, then Q lies on the opposite side of P from A. Thus, $\overrightarrow{PQ}$ has the same direction as $\overrightarrow{PA}$ if θ is acute, and it has the opposite direction from $\overrightarrow{PA}$ if θ is obtuse.

$\sphericalangle PQB$ is a right angle. ΔPQB is a right triangle; it has hypotenuse $\overline{PB}$ and legs $\overline{PQ}$ and $\overline{QB}$, and we denote the lengths of these sides as PB, PQ, and QB, respectively. $PB = b$.

We shall now argue that $\overrightarrow{PQ}$ represents $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$. To establish this, we must demonstrate that $\overrightarrow{PQ}$ has the appropriate length and direction; i.e., we must show that $PQ = \left|\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}\right|$ and that $\overrightarrow{PQ}$ and $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$ have the same direction.

$\cos\sphericalangle QPB = \frac{adj}{hyp} = \frac{PQ}{PB} = \frac{PQ}{b}$, so $PQ = b\cos\sphericalangle QPB$. If θ is acute, then $\sphericalangle QPB = \theta$, so $PQ = b\cos\theta$. If θ is obtuse, then $\sphericalangle QPB = \pi - \theta$, so $PQ = b\cos(\pi - \theta) = -b\cos\theta$. In both cases, we have $PQ = |b\cos\theta|$. Substituting $\frac{\mathbf{a}\cdot\mathbf{b}}{ab}$ in place of $\cos\theta$, we obtain $PQ = \left|b\frac{\mathbf{a}\cdot\mathbf{b}}{ab}\right| = |\frac{\mathbf{a}\cdot\mathbf{b}}{a}|$.

The unit vector in the direction of $\mathbf{a}$ is $\frac{\mathbf{a}}{a}$. $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$ is equal to $\frac{\mathbf{a}\cdot\mathbf{b}}{a}\frac{\mathbf{a}}{a}$, so its magnitude is $|\frac{\mathbf{a}\cdot\mathbf{b}}{a}\frac{\mathbf{a}}{a}| = |\frac{\mathbf{a}\cdot\mathbf{b}}{a}||\frac{\mathbf{a}}{a}| = |\frac{\mathbf{a}\cdot\mathbf{b}}{a}| = PQ$. Thus, $PQ = \left|\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}\right|$.

When θ is acute, $\mathbf{a}\cdot\mathbf{b} > 0$, so $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$ is a positive scalar multiple of $\mathbf{a}$ and therefore has the same direction as $\mathbf{a}$, which is the same direction as $\overrightarrow{PA}$. When θ is obtuse, $\mathbf{a}\cdot\mathbf{b} < 0$, so $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$ is a negative scalar multiple of $\mathbf{a}$ and therefore has the opposite direction from $\mathbf{a}$, which is the same direction as $\overrightarrow{PA}$. Thus, in both cases, $\overrightarrow{PQ}$ and $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$ have the same direction.

Ergo, $\overrightarrow{PQ}$ represents $\frac{\mathbf{a}\cdot\mathbf{b}}{a^2}\mathbf{a}$.

Summary: Let θ be the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$.

1. If $\theta = \frac{\pi}{2}$, then $proj_{\mathbf{a}}\mathbf{b} = \mathbf{0}$.
2. If $\theta = 0$ or $\theta = \pi$, then $proj_{\mathbf{a}}\mathbf{b} = \mathbf{b}$.
3. If θ is acute, then $proj_{\mathbf{a}}\mathbf{b}$ is the vector in the same direction as $\mathbf{a}$ whose length is $\frac{\mathbf{a}\cdot\mathbf{b}}{a}$.
4. If θ is obtuse, then $proj_{\mathbf{a}}\mathbf{b}$ is the vector in the opposite direction from $\mathbf{a}$ whose length is the absolute value of $\frac{\mathbf{a}\cdot\mathbf{b}}{a}$.

Actually, the case where $proj_{\mathbf{a}}\mathbf{b} = \mathbf{b}$ (case 2 in the above summary) does not need to be listed separately; it can be merged with cases 3 and 4 above. Thus, our summary can be condensed as follows:

1. If $\theta = \frac{\pi}{2}$, then $proj_{\mathbf{a}}\mathbf{b} = \mathbf{0}$.
2. If $\theta < \frac{\pi}{2}$, then $proj_{\mathbf{a}}\mathbf{b}$ is the vector in the same direction as $\mathbf{a}$ whose length is $\frac{\mathbf{a}\cdot\mathbf{b}}{a}$. (In the special case where $\theta = 0$, we get $\mathbf{b}$.)
3. If $\theta > \frac{\pi}{2}$, then $proj_{\mathbf{a}}\mathbf{b}$ is the vector in the opposite direction from $\mathbf{a}$ whose length is the absolute value of $\frac{\mathbf{a}\cdot\mathbf{b}}{a}$. (In the special case where $\theta = \pi$, we get $\mathbf{b}$.)

In *all* cases, the magnitude of $proj_{\mathbf{a}}\mathbf{b}$ is $|\frac{\mathbf{a}\cdot\mathbf{b}}{a}|$, so we may think of $\frac{\mathbf{a}\cdot\mathbf{b}}{a}$ as the **signed magnitude** of $proj_{\mathbf{a}}\mathbf{b}$. In other words, $\frac{\mathbf{a}\cdot\mathbf{b}}{a}$ is the magnitude of $proj_{\mathbf{a}}\mathbf{b}$ when $\theta \leq \frac{\pi}{2}$, and it is the *negative* of the magnitude when $\theta > \frac{\pi}{2}$.

$\frac{\mathbf{a}\cdot\mathbf{b}}{a}$ is known as the **component** of $\mathbf{b}$ **along** $\mathbf{a}$. It is also known as the **scalar projection** of $\mathbf{b}$ **onto** $\mathbf{a}$. It is denoted $comp_{\mathbf{a}}\mathbf{b}$.

Note that $proj_{\mathbf{a}}\mathbf{b}$ is a vector, whereas $comp_{\mathbf{a}}\mathbf{b}$ is a scalar, and $proj_{\mathbf{a}}\mathbf{b} = (comp_{\mathbf{a}}\mathbf{b})\frac{\mathbf{a}}{a}$. In other words, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is equal to the component of $\mathbf{b}$ along $\mathbf{a}$ times the unit vector in the direction of $\mathbf{a}$. We also have $proj_{\mathbf{a}}\mathbf{b} = \frac{comp_{\mathbf{a}}\mathbf{b}}{a}\mathbf{a} = \frac{1}{a}(comp_{\mathbf{a}}\mathbf{b})\mathbf{a}$.

Since $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab}$, $b\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{a} = comp_{\mathbf{a}}\mathbf{b}$.

Example 16: Let $\mathbf{a} = < 4,3 >$ and $\mathbf{b} = < 2,4 >$. $a = 5$ and $\mathbf{a} \cdot \mathbf{b} = 20$, so $comp_{\mathbf{a}}\mathbf{b} = 4$ and $proj_{\mathbf{a}}\mathbf{b} = \frac{4}{5}\mathbf{a} = \frac{4}{5} < 4,3 > = < \frac{16}{5}, \frac{12}{5} >$. In terms of the geometry discussed above, if we draw $\mathbf{a}$ and $\mathbf{b}$ in standard position, then $P = (0,0)$, $A = (4,3)$, $B = (2,4)$, the line $\overleftrightarrow{PA}$ is $y = \frac{3}{4}x$, the line ℓ is $y = -\frac{4}{3}x + \frac{20}{3}$, and the point Q is $(\frac{16}{5}, \frac{12}{5})$ or $(3.2, 2.4)$.

Example 17: Let $\mathbf{a} = < -2,3,1 >$ and $\mathbf{b} = < 1,1,2 >$. $a = \sqrt{14}$ and $\mathbf{a} \cdot \mathbf{b} = 3$, so $comp_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{14}}$ and $proj_{\mathbf{a}}\mathbf{b} = \frac{1}{\sqrt{14}} \frac{3}{\sqrt{14}}\mathbf{a} = \frac{3}{14}\mathbf{a} = \frac{3}{14} < -2,3,1 > = < \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} >$.

If $\mathbf{a}$ is a unit vector, then $a = 1$, and our formulas simplify as follows:

- $comp_{\mathbf{a}}\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$
- $proj_{\mathbf{a}}\mathbf{b} = (comp_{\mathbf{a}}\mathbf{b})\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{a}$

These special formulas will be very important later on.

Suppose that a particle in either two-dimensional space or three-dimensional space moves in a straight line from point C to point D (these being two distinct points). The directed line segment $\overrightarrow{CD}$ is referred to as the particle's **displacement vector**, which may be denoted $\mathbf{d}$. Its magnitude, d, is the **distance** traversed by the particle. Suppose a constant force $\mathbf{f}$ is exerted on the particle along this path, with f being the magnitude of the force. If $\mathbf{f}$ has the same direction as $\mathbf{d}$, then the *work* done is the product fd. This is a special case of a more general formula. More generally, $\mathbf{f}$ may not have the same direction as $\mathbf{d}$. The work done depends upon the component of $\mathbf{f}$ along $\mathbf{d}$, $comp_{\mathbf{d}}\mathbf{f} = \frac{\mathbf{d} \cdot \mathbf{f}}{d} = f\cos\theta$ (where θ is the angle between $\mathbf{d}$ and $\mathbf{f}$). We also refer to this as the **component of force in the direction of motion**. To be precise, the **work** done, denoted W, is equal to the *product* of the component of force in the direction of motion and the distance traversed, i.e., $W = (comp_{\mathbf{d}}\mathbf{f})d = \mathbf{f} \cdot \mathbf{d} = fd\cos\theta$.

Note that the work done has the potential to be positive or negative or zero. Specifically, it is *positive* if $\theta < \frac{\pi}{2}$, it is *zero* if $\theta = \frac{\pi}{2}$, and it is *negative* if $\theta > \frac{\pi}{2}$.

If the magnitude of force is measured in *newtons* and distance is measured in *meters*, then work is measured in *newton-meters*, also known as *joules*.

Example 18: A particle in two-dimensional space moves horizontally a distance of 100 meters as a constant force is exerted upon it. The magnitude of the force is 70 newtons, and the angle between the force and the direction of motion is 35°. The work done is thus $(70)(100)\cos 35^\circ$, which is $5,734$ joules, rounded to the nearest whole number. (In this situation, we have $\mathbf{d} = < 100, 0 >$. $\mathbf{f}$ has direction angles $\alpha = 35^\circ$ and $\beta = 55^\circ$, so $\mathbf{f} = 70 < \cos 35^\circ, \cos 55^\circ > = < 70\cos 35^\circ, 70\cos 55^\circ > \approx < 57.34, 40.15 >$, and $\mathbf{f} \cdot \mathbf{d} =$ $< 70\cos 35^\circ, 70\cos 55^\circ > \cdot < 100, 0 > \approx < 57.34, 40.15 > \cdot < 100, 0 > = 5,734.$)

Example 19: A particle in three-dimensional space moves linearly from the point $(2,1,0)$ to the point $(4,6,2)$ as a constant force $\mathbf{f} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ is exerted upon it. Here we have $\mathbf{d} = < 2,5,2 >$ and $\mathbf{f} = < 3,4,5 >$, so the work done is $\mathbf{f} \cdot \mathbf{d} = < 3,4,5 > \cdot < 2,5,2 >$ $= 6 + 20 + 10 = 36$.